EE C145B / BioE C165 Spring 2004: Practice midterm

Questions marked with an asterisk are more difficult. It may be prudent to answer them towards the end of the examination period.

Total points: 205 + 35 bonus.

Question 1 (20)

You are given the following matrix that represents the gray levels of an image:

$$\begin{bmatrix} 2.0 & 6.0 & 4.0 & 1.0 & 9.0 \\ 7.0 & 0.0 & 7.0 & 3.0 & 8.0 \\ 9.0 & 10.0 & 6.0 & 1.0 & 3.0 \\ 3.0 & 9.0 & 3.0 & 4.0 & 0.0 \\ 6.0 & 7.0 & 9.0 & 3.0 & 0.0 \end{bmatrix}$$

Equalize the histogram of this image. Show all steps. Use ten histogram bins spanning the range $[0, 10]$. Your solution should be close to or equal to:

$$\begin{bmatrix} 2.5 & 5.5 & 4.5 & 1.5 & 9.5 \\ 6.5 & 0.5 & 7.5 & 3.5 & 7.5 \\ 8.5 & 9.5 & 6.5 & 1.5 & 4.5 \\ 2.5 & 8.5 & 3.5 & 5.5 & 0.5 \\ 5.5 & 7.5 & 9.5 & 3.5 & 1.5 \end{bmatrix}$$

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Question 2 (20)

The 8-element vector

$$\begin{bmatrix} 3, & -4\jmath, & 0, & 1+\jmath, & 1, & 1-\jmath, & 0, & 4\jmath \end{bmatrix} \times 8 \tag{1}$$

corresponds to the DFT of a signal.

1. Draw the sinusoidal components of this signal on separate but identical axes.
2. What operation must be performed on these five signals to yield the original signal?

Question 3 (15)

Explain, with the aid of a diagram if necessary, the reason why a square image with square pixels has its greatest sampling rate along the diagonal.

Question 4 (10)

Explain the difference between image enhancement and restoration.

Question 5 (15)

Find the value of x that minimizes:

$$\left\| \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} + \begin{bmatrix} 3x \\ 2x+5 \\ 0 \end{bmatrix} \right\|$$

where y is a physical measurement you have made. Does this value depend on your measurement?

Question 6 (10)

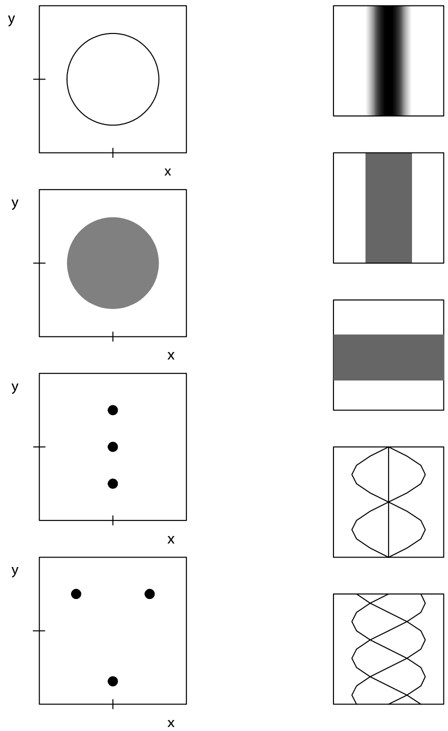
Find the value of x that minimizes:

$$\left\| \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} + \begin{bmatrix} 3x \\ 2x \\ 10x \end{bmatrix} \right\|$$

where y is a physical measurement you have made. Does this value depend on your measurement?

Question 7 $(5+5+5+5)$

The column on the right contains possible sinograms of the distributions in the left column. Match the elements of the two columns.



Question 8 $(10 + 10)$

The following matrix contains a cropped area of a larger image:

$$\begin{bmatrix} 2 & 5 & 3 & 0 & 0 \\ 8 & 7 & 1 & 8 & 2 \\ 5 & 4 & 3 & 0 & 5 \\ 1 & 2 & 9 & 11 & 2 \\ 6 & 5 & 3 & 8 & 4 \\ 1 & 0 & 6 & 7 & 2 \end{bmatrix}$$

Consider the element at row 3 and column 3 of this matrix.

1. What would a 4×4 median filter replace this value with? Specify your conventions.

2. What would a 3×3 10th-percentile order statistics filter replace this value with? (Hint: The median filter is a 50th-percentile order statistics filter)

Question 9 $(15 + 5)$

The following is the kernel of a Gaussian low-pass filter.

1. Convert it to a Gaussian high-pass filter and write out the matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} /16$$

2. Give an intuitive explanation, based on the kernel values, as to why the one filter is high-pass and the other is low-pass.

Question 10 (20)

Prove the 2D projection slice theorem (in vector or scalar form).

Question 11 $(20 + 10$ bonus)

The eigenvalue decomposition of a matrix $\mathbf{X}$ can be written as:

$$\mathbf{XZ} = \mathbf{Z\Lambda}$$

where $\mathbf{Z}$ contains as columns the eigenvectors of $\mathbf{X}$, and $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{X}$. The eigenvalues are sorted in descending order along the diagonal of $\mathbf{\Lambda}$. The corresponding eigenvectors are arranged in $\mathbf{Z}$ in the same order.

The singular value decomposition of $\mathbf{F}$ is given by:

$$\mathbf{F} = \mathbf{USV}^T$$

Given that the eigenvectors of $\mathbf{F}^T\mathbf{F}$ are the same at the right singular vectors of $\mathbf{F}$:

1. Show that the eigenvalues of $\mathbf{F}^T\mathbf{F}$ are the squares of the corresponding singular values of $\mathbf{F}$.

2. * In Matlab you store the number:

   $$n = 0.141592653589793115997963468544185161590576171875$$

   in an 8 byte floating point variable and cube it. You then take the cube root of the result and get:

   $$0.14159265358979314375353908417309867218136787414550781 25$$

   With this example and Part One of this question in mind, give a reason why we prefer to calculate the pseudoinverse using the SVD rather than by direct calcluation of $(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T$?

Question 12 (15)

Show diagramatically, how 2D ramp filtering of a backprojection image may be performed using analog components only.

Question 13 (25 bonus)

* In class, we derived the pseudoinverse by formulating the sum of squared residuals:

$$C = \sum_{m=1}^{M} r_m^2 = \sum_{m=1}^{M} \left[\sum_{n=1}^{N} f_n^m(x_m)\, \theta_n - y_m \right]^2$$

and minimizing it by taking its derivative with respect to each of the parameters θ_n, setting these equations equal to zero, and solving for $\boldsymbol{\theta}$. It is much easier to derive the pseudoinverse from the vector formulation.
Starting with:

$$C = \|\mathbf{y} - \mathbf{F}\boldsymbol{\theta}\|^2 = (\mathbf{y} - \mathbf{F}\boldsymbol{\theta})^T (\mathbf{y} - \mathbf{F}\boldsymbol{\theta})$$

solve for the solution $\hat{\boldsymbol{\theta}}$ that minimizes C. Hint:

$$\frac{d}{d\mathbf{x}} \mathbf{Q}\,\mathbf{x} = \mathbf{Q}^T$$

$$\frac{d}{d\mathbf{x}} \mathbf{q}\,\mathbf{x} = \mathbf{q}^T$$

$$\frac{d}{d\mathbf{x}} \mathbf{x}^T\, \mathbf{Q}\,\mathbf{x} = 2\,\mathbf{Q}^T \mathbf{x}$$

Working backwards from the solution may help.