EECS C145B / BioE C165: Image Processing and Reconstruction Tomography

Information sheet for examinations

May 7, 2003

Discrete convolution in 1D:

$$g[n] = \sum_{n'=-\infty}^{\infty} f[n-n']h[n']$$

Discrete convolution in 2D:

$$g[n,m] = \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} f[m-m', n-n']\, h[m', n']$$

Forward DFT (analysis):

$$G[k] = \begin{cases} \sum_{n=0}^{N-1} g[n] \mathrm{e}^{-\jmath 2\pi kn/N}, & 0 \le k \le N-1, \\ 0 & \text{otherwise} \end{cases}$$

Inverse DFT (synthesis):

$$g[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} G[k]\, \mathrm{e}^{\jmath 2\pi kn/N}, & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$$

Fourier transform: (analysis)

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\jmath \omega x} dx$$

Inverse Fourier transform: (synthesis)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{\jmath \omega x} d\omega$$

$$\mathrm{comb}(x/X, y/Y) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(x-kX,\, y-lY)$$

$$\mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

$$\mathrm{rect}\left(\frac{x}{X}\right) = \begin{cases} 1, & |x| < X/2 \\ 0, & |x| > X/2 \end{cases}$$

$$\mathrm{rect}\left(\frac{x}{X}, \frac{y}{Y}\right) \triangleq \mathrm{rect}\left(\frac{x}{X}\right) \mathrm{rect}\left(\frac{y}{Y}\right)$$

$$\mathrm{rect}\left(\frac{r}{R}\right) \triangleq \begin{cases} 1, & |r| < R \\ 0, & |r| > R \end{cases}$$

FT properties

Duality:

$$\begin{aligned} g(x) &\rightleftharpoons f(\omega) \\ f(x) &\rightleftharpoons 2\pi g(-\omega) \end{aligned}$$

$$\sum_{k=-\infty}^{\infty} \delta(x - kX) \rightleftharpoons \sum_{k=-\infty}^{\infty} \delta(\omega - k\,2\pi/X)$$

$$f(x) * h(x) \rightleftharpoons F(\omega)H(\omega)$$

$$f(x)h(x) \rightleftharpoons \frac{1}{2\pi}\Big(F(\omega) * H(\omega)\Big)$$

$$f_s(x) = \sum_{k=-\infty}^{\infty} \delta(x - kX) f(x)$$

$$F_s(\omega) = \frac{2\pi}{X} \sum_{k=-\infty}^{\infty} \delta(\omega - k\,2\pi/X) * F(\omega)$$

Forward transform: (analysis)

$$F(w_1, w_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, \mathrm{e}^{-\jmath(w_1 x_1 + w_2 x_2)} dx_1 dx_2$$

Inverse transform: (synthesis)

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_1, \omega_2)\, \mathrm{e}^{\jmath(\omega_1 x_1 + \omega_2 x_2)} d\omega_1 d\omega_2$$

Forward transform: (analysis)

$$F(\boldsymbol{\omega}) = \int_{\Re^n} f(\mathbf{x})\, \mathrm{e}^{-\jmath\,\boldsymbol{\omega}\cdot\mathbf{x}} d\mathbf{x}$$

Inverse transform: (synthesis)

$$f(\mathbf{x}) = (2\pi)^{-n} \int_{\Re^n} F(\boldsymbol{\omega})\, \mathrm{e}^{\jmath\,\boldsymbol{\omega}\cdot\mathbf{x}} d\boldsymbol{\omega}$$

Spectrum	$\lvert F(u,v)\rvert = \sqrt{R(u,v)^2 + I(u,v)^2}$
(magnitude)	$R(u,v)$: real part of $F(u,v)$
	$I(u,v)$: imaginary part of $F(u,v)$
Phase	$\phi(u,v) = \arctan\left(\frac{I(u,v)}{R(u,v)}\right)$
Conjugate symmetry	$F(u,v) = F^*(-u,-v)$
	$\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Duality	If $g(x,y) \rightleftharpoons f(u,v)$
	then $f(x,y) \rightleftharpoons g(-u,-v)$
Translation	$f(x \pm x_0, y \pm y_0) \rightleftharpoons F(u,v)\mathrm{e}^{\pm\jmath 2\pi(ux_0 + vy_0)}$
	$f(x,y)\mathrm{e}^{\mp\jmath 2\pi(u_0 x + v_0 y)} \rightleftharpoons F(u \pm u_0, v \pm v_0)$
Rotation	$f(r, \theta + \theta_0) \rightleftharpoons F(\rho, \phi + \theta_0)$
Convolution	$f(x,y) * h(x,y) \rightleftharpoons F(u,v)H(u,v)$

Space domain	Frequency domain
$\delta(x,y)$	1
$\delta(x-x_0, y-y_0)$	$\mathrm{e}^{\pm \jmath 2\pi x_0 u}\mathrm{e}^{\pm \jmath 2\pi y_0 v}$
$\mathrm{rect}\left(\frac{x}{A}, \frac{y}{B}\right)$	$AB\,\mathrm{sinc}(Au, Bv) = AB\,\frac{\sin(\pi u A)}{\pi u A}\frac{\sin(\pi v B)}{\pi v B}$
$\mathrm{rect}(r/R),\ r=\sqrt{x^2+y^2}$	$R\,\mathrm{jinc}(R\rho),\ \rho=\sqrt{u^2+v^2}$
$\mathrm{comb}(x/X, y/Y)$	$\mathrm{XY}\,\mathrm{comb}(uX, vY)$
$\cos(2\pi(u_o x + v_0 y))$	$\frac{1}{2}(\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0))$
$\sin(2\pi(u_o x + v_0 y))$	$\jmath\frac{1}{2}(\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0))$
$\mathrm{e}^{-\pi(x^2+y^2)}$	$\mathrm{e}^{-\pi(u^2+v^2)}$

$$\begin{aligned}
\sin(\alpha)\cos(\beta) &= \frac{1}{2}\Big[\sin(\alpha-\beta)+\sin(\alpha+\beta)\Big] \\
\sin(\alpha)\sin(\beta) &= \frac{1}{2}\Big[\cos(\alpha-\beta)-\cos(\alpha+\beta)\Big] \\
\cos(\alpha)\cos(\beta) &= \frac{1}{2}\Big[\cos(\alpha-\beta)+\cos(\alpha+\beta)\Big]
\end{aligned}$$

$$\mathrm{e}^{\pm\jmath\theta} = \cos(\theta) \pm \jmath\sin(\theta)$$

$$F[k,l] = F_s(u,v) \times \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta(u - ku_s/N, v - lv_s/M)$$

$$f_p(x,y) = f_s(x,y) * \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x - nN/u_s, y - mM/v_s)$$

$$f_s(x,y) = f(x,y) \times \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta(x - k/u_s, y - l/v_s)$$

$$F_s(u,v) = F(u,v) * \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta(u - ku_s,\ y - lv_s)$$

$$h(x,y) = \frac{1}{2\pi\sigma^2}\,\mathrm{e}^{-(x^2+y^2)/\sigma^2}$$

$$\boldsymbol{\theta}_{\mathrm{LS}} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{y}$$

$$\boldsymbol{\theta}_{\mathrm{LS}} = \mathbf{F}^{+}\mathbf{y}$$

$$\mathbf{F} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

$$\mathbf{F} = \sum_{n=1}^{N} \sigma_n \mathbf{u}_n \mathbf{v}_n^T$$

$$I = I_0\,\mathrm{e}^{-\int_L \mu(\mathbf{x})\,du}$$

$$I = I_0\,\mathrm{e}^{-\sum_{k=1}^{K}\mu_k\,l_k}$$

$$p = \ln\frac{I_0}{I}$$

$$
\begin{aligned}
p \triangleq p(\boldsymbol{\theta}, s) &= \int_L \mu(\mathbf{x})\, du \\
&= \int_{\mathbf{x}\cdot\boldsymbol{\theta}=s} \mu(\mathbf{x})\, d\mathbf{x}
\end{aligned}
$$

$$
p(\boldsymbol{\theta}, s) = \mathbf{R}f(\mathbf{x}) = \int_{\mathbf{x}\cdot\boldsymbol{\theta}=s} f(\mathbf{x})\, d\mathbf{x} = \int_{\Re^N} f(\mathbf{x})\, \delta(\mathbf{x}\cdot\boldsymbol{\theta} - s)\, d\mathbf{x}
$$

$$
\begin{aligned}
p(\boldsymbol{\theta}, s) &= \mathbf{R}f(x, y) = \int_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\cdot\boldsymbol{\theta}=s} f(x, y)\, dx\, dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, \delta\big(x\, \cos(\theta) + y\, \sin(\theta) - s\big) dx\, dy
\end{aligned}
$$

Property	Distribution	Radon transform $\mathbf{R}f$
Linearity	$f(\mathbf{x}) = \sum_i f_i(\mathbf{x})$	$\mathbf{R}f(\mathbf{x}) = \sum_i \mathbf{R}f_i(\mathbf{x})$
Limited support	$f(\mathbf{x}) = 0,\ \lvert x_i\rvert > D/2$	$\mathbf{R}f(\mathbf{x}) = 0,\ \lvert s\rvert > D\sqrt{N}/2$
Symmetry	$f(\mathbf{x})$	$\mathbf{R}f(\mathbf{x}) = p(\boldsymbol{\theta}, s) = p(-\boldsymbol{\theta}, -s)$
	$f(x, y)$	$p(\boldsymbol{\theta}, s) = p_\theta(s) = p_{\theta\pm\pi}(-s)$
Periodicity	$f(x, y)$	$p_\theta(s) = p_{\theta\pm 2k\pi}(s),\ k \in \mathcal{Z}$
Translation	$f(\mathbf{x} - \mathbf{x}_0)$	$p(\boldsymbol{\theta}, s - \mathbf{x}_0\cdot\boldsymbol{\theta})$
	$f(x - x_0, y - y_0)$	$p(\boldsymbol{\theta}, s - x_0\cos(\theta) - y_0\sin(\theta))$
Rotation	$f(r, \mathbf{T}\boldsymbol{\phi})$	$p(\mathbf{T}\boldsymbol{\theta}, s),\ \mathbf{T}$: rotation matrix
	$f(r, \phi + \theta_0)$	$p_{\theta+\theta_0}(s)$
Scaling	$f(a\mathbf{x})$	$\frac{1}{\lvert a\rvert}p(\boldsymbol{\theta}, as)$
Mass conservation	$M = \int_{\Re^N} f(\mathbf{x})\, d\mathbf{x}$	$M = \int_{-\infty}^{\infty} p(\boldsymbol{\theta}, s)\, ds$

$$
\int_{\Re^2} f(x, y)\, dx\, dy = \int_S \int_0^\infty f(r, \boldsymbol{\theta})\, \lvert\mathbf{J}\rvert\, dr\, d\theta = \int_S \int_0^\infty f(r, \boldsymbol{\theta})\, r\, dr\, d\theta
$$

$$
F(\boldsymbol{\xi}) = \int_{\Re^2} f(\mathbf{x})\, \mathrm{e}^{-\jmath 2\pi\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, \mathrm{e}^{-\jmath 2\pi(ux+vy)} dx\, dy = F(u, v)
$$

$$
P_\theta(w) = \int_{-\infty}^{\infty} p(\boldsymbol{\theta}, s)\, \mathrm{e}^{-\jmath 2\pi w s}\, ds
$$

$$
f_{\mathrm{BP}}(\mathbf{x}) = \mathrm{BP}\{p(\boldsymbol{\theta}, s)\} \triangleq \int_0^\pi p(\boldsymbol{\theta}, s)\, d\boldsymbol{\theta}
$$

$$
f_{\mathrm{BP}}(\mathbf{x}) = \frac{1}{I}\sum_{i=1}^{I} p(\boldsymbol{\theta}_i, s = \mathbf{x}\cdot\boldsymbol{\theta}_i)
$$

$$
\begin{aligned}
f(\mathbf{x}) &= \int_0^\pi \mathcal{F}_1^{-1}\Big\{2\pi\jmath\, w P_{\boldsymbol{\theta}}(w)\Big\} * \mathcal{F}_1^{-1}\Big\{\frac{\mathrm{sgn}(w)}{2\pi\jmath}\Big\} d\theta \\
&= \int_0^\pi \frac{\partial p(\boldsymbol{\theta}, s)}{\partial s} * \frac{1}{2\pi^2 s}\, d\theta
\end{aligned}
$$

$$
J = \left\lceil \frac{D}{\Delta s} \right\rceil = \lceil 2D\, \rho_{\max} \rceil
$$

$$
I = \left\lceil \frac{\pi}{\Delta\theta} \right\rceil = \lceil \pi D\, \rho_{\max} \rceil
$$

$$p_{ij} = \lambda_{ij} + \epsilon$$

$$f_E(\epsilon) = \frac{1}{\sqrt{2\pi}\sigma_\epsilon}\, \mathrm{e}^{-\frac{1}{2\sigma_\epsilon^2}\,\epsilon^2},$$

$$P(\epsilon_1 < E \leq \epsilon_2) = \int_{\epsilon_1}^{\epsilon_2} f_E(\omega)\, d\omega$$

$$P(A \cap B) = P(A) \times P(B)$$

$$P(\cap_{i=1}^{N} A_i) = \prod_{i=1}^{N} P(A_i)$$

$$f_E(\boldsymbol{\epsilon}) \quad = \quad \frac{1}{(2\pi)^{IJ/2}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\epsilon}}$$

$$f_{\mathcal{P}|\Lambda}(\mathbf{p}, \boldsymbol{\lambda}) \quad = \quad \frac{1}{(2\pi)^{IJ/2}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\, \mathrm{e}^{-\frac{1}{2}(\mathbf{p}-\boldsymbol{\lambda})^T\boldsymbol{\Sigma}^{-1}(\mathbf{p}-\boldsymbol{\lambda})}$$

$$f_{\mathcal{P}|\mathcal{U}}(\mathbf{p}, \boldsymbol{\mu}) \quad = \quad \frac{1}{(2\pi)^{IJ/2}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\, \mathrm{e}^{-\frac{1}{2}(\mathbf{p}-\mathbf{F}\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{p}-\mathbf{F}\boldsymbol{\mu})}$$

$$\ell(\boldsymbol{\mu}) \quad = \quad -\ln\left[\frac{1}{(2\pi)^{IJ/2}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\right] + \frac{1}{2}\mathbf{p}^T\boldsymbol{\Sigma}^{-1}\mathbf{p} - 2\,\mathbf{p}^T\mathbf{F}\boldsymbol{\mu} + \boldsymbol{\mu}^T\mathbf{F}^T\boldsymbol{\Sigma}^{-1}\mathbf{F}\boldsymbol{\mu}$$

$$\boldsymbol{\mu}_{\mathrm{WLS}} = \left(\mathbf{F}^T\boldsymbol{\Sigma}^{-1}\mathbf{F}\right)^{-1}\mathbf{F}^T\boldsymbol{\Sigma}^{-1}\mathbf{p}$$

$$P(\Delta T_i \geq s + t \,|\, \Delta T_i \geq s) = P(\Delta T_i \geq t)$$

$$p(t) = \beta\, \mathrm{e}^{-\beta t}$$

$$P(N(t) = k) = \mathrm{e}^{-\beta t}\frac{(\beta t)^k}{k!}$$

$$P(N = k) = \mathrm{e}^{-\lambda}\frac{(\lambda)^k}{k!}$$

$$\ell(\boldsymbol{\mu}) \quad = \quad -\sum_{i=1}^{I}\sum_{j=1}^{J} -\lambda_{ij}(\boldsymbol{\mu}) + p_{ij}\ln(\lambda_{ij}(\boldsymbol{\mu})) - \ln(p_{ij}!)$$

$$N_e > \frac{2\pi D}{d}$$

$${}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + \beta^- + \nu + Q$$

$${}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}Y + \beta^+ + \nu + Q$$

$$ {}^{A}_{Z}X + e^- \rightarrow {}^{A}_{Z-1}Y + \nu + Q $$

$$ \frac{d\mathbf{m_p}}{dt} = \gamma \mathbf{m_p} \times \mathbf{B_0} $$

$$ \omega_0 = -\gamma \mathbf{B}_0 $$

$$ \alpha = \gamma_0 \int_0^{\tau_p} H_1(t) dt $$

$$ \frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B} $$

$$ \frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1} $$

$$ M_z(t) = M_0 \left(1 - \mathrm{e}^{-t/T_1}\right) $$

$$ \begin{aligned} \frac{dM_x}{dt} &= -\frac{M_x}{T_2} \\ \frac{dM_y}{dt} &= -\frac{M_y}{T_2} \end{aligned} $$

$$ \begin{aligned} M_x(t) &= M_x(0)\,\mathrm{e}^{-t/T_2} \\ M_y(t) &= M_y(0)\,\mathrm{e}^{-t/T_2} \end{aligned} $$

$$ M_{xy}(t) = M_0\,\mathrm{e}^{-t/T_2} $$

$$ \frac{1}{T_2^*} = \frac{1}{T_2} + \gamma \Delta B_0/2 $$

$$ \mathbf{I} = \hbar\sqrt{I(I+1)} $$

$$ E = -\gamma \hbar\, m_I B_0 $$

$$ \Delta E = E_2 - E_1 = -\gamma\hbar\Big[-\frac{1}{2} - \frac{1}{2}\Big] B_0 = \gamma \hbar B_0 $$

$$ \hbar\,\omega_0 = \gamma B_0\,\hbar $$

$$ s(t) = S\cos(\omega_0 t + \phi)\,\mathrm{e}^{-t/T_2^*} $$

$$ M_z(TR) = M_0\big(1 - \mathrm{e}^{-TR/T_1})\big) $$

$$ M_z(nTR) = M_0\big(1 - \mathrm{e}^{-TR/T_1}\big), \qquad n \in \mathcal{Z}, n > 0 $$

$$S \propto M_0\big(1 - \mathrm{e}^{-T_R/T_1}\big)\big(\mathrm{e}^{-T_E/T_2}\big)$$

$$\omega(x) = \gamma(B_0 + xG_x)$$

$$\frac{dM_{xy}}{dt} = \left(\jmath\,\omega_0 - \frac{1}{T_2} - \jmath\,\gamma\,B_g(\mathbf{r})\right)M_{xy}$$

$$M_{xy} = M_y + jM_x$$

$$\begin{aligned} B_g(\mathbf{r}) &= (x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z)\cdot\big[G_x(t)\,\mathbf{e}_x + G_y(t)\,\mathbf{e}_y + G_z(t)\,\mathbf{e}_z\big] \\ &= \mathbf{r}\cdot\mathbf{G}(t). \end{aligned}$$

$$M_{xy}(\mathbf{r},t) = M_0\,\rho(\mathbf{r})\,\mathrm{e}^{-\jmath\gamma\mathbf{r}\cdot\int_0^t \mathbf{G}(\tau)\,d\tau}$$

$$S(t) = M_0\int \rho(\mathbf{r})\,\mathrm{e}^{-\jmath\gamma\mathbf{r}\cdot\int_0^t \mathbf{G}(\tau)d\tau}d\mathbf{r}.$$

$$S(t) = M_0\iint \rho(x,y)\,\mathrm{e}^{-\jmath\gamma(x\,\mathbf{e}_x+y\,\mathbf{e}_y)\cdot\int_0^t\big[G_x(\tau)\,\mathbf{e}_x+G_y(\tau)\,\mathbf{e}_y\big]d\tau}dx\,dy.$$

$$\begin{aligned} S(t_x,T_y) &= M_0\iint \rho(x,y)\,\mathrm{e}^{-\jmath\gamma(xG_xt_x+yG_yT_y)}dx\,dy \\ &= M_0\iint \rho(x,y)\,\mathrm{e}^{-\jmath\gamma(xu+yv)}dx\,dy \end{aligned}$$

$$\Delta f = f - f_0 = \frac{\pm 2f_0 v\cos\theta}{c}$$